String harmonics in detail: a pedagogical resource

1. Overview. 2. The overtone series; intervals. 3. String length and frequency. 4. Natural harmonics. 5. Artificial harmonics.

1. Overview

String harmonics are commonplace in solo, chamber music, and orchestral literature. The finger of the left hand, rather than stop the string, touches it only firmly enough to create a "node," a point on the string that does not itself vibrate but which permits vibration on either side. Many sources clearly present the basic information about harmonics, including those most frequently employed (see Table 1) and the systems of notation that have now become standard.[1]
In principle, no composer, conductor, or instrumentalist should have any difficulty with harmonics and their notation. Yet confusion persists, due largely to two important factors:

- Even the best standard references provide little information about the physical and mathematical issues behind the production of harmonics. Without learning the theory behind the practice, musicians have difficulty remembering what they know. Once their memory fails them, they lack the tools to figure out harmonics on their own. Furthermore, if a composer has asked for (or wishes to ask for) a harmonic that is not one of the "standards," it is unlikely that the performers will be able to figure out what is intended.

[1] Notation of harmonics is not discussed in detail, since general familiarity is assumed. An overview is provided by Walter Piston, *Orchestration* (New York: W. W. Norton, 1955), 29-31.

• The standard notation system is intrinsically confusing. The location of the notehead instructs the performer where to place his or her finger, while the open diamond indicates that the string should be not closed, but only touched. Visually, a relationship is implied between the notated pitch and the sounding pitch. In practice, however, this relationship is only sometimes evident.

touching the string here	produces a harmonic sounding this far above the open string
at the octave	one octave
at the perfect fifth	one octave plus a perfect fifth (a perfect 12th)
at the perfect fourth	two octaves
at the major third	two octaves plus a major third
at the major sixth	two octaves plus a major third

Table 1. The most frequently employed natural harmonics.

Thus, even at the elementary level of Table 1, the confusion inherent in discussing harmonics becomes apparent. What does it mean to say, "touching the string *at the major sixth?*" It is a shorthand that really means, "touching the string at the point which, if the finger were to be depressed fully, would result in the string vibrating at a pitch a major sixth higher than if it were open." But what *is* this point on the string? Why does it raise the tone by a major sixth in one case, and by two octaves and a major third in the other?

These questions may seem mysterious or arcane, but their answers to not require anything beyond high-school algebra, and mastery of these issues can be immensely valuable to anyone who aims to write, read, or interpret harmonics.

2. The Overtone Series; Intervals

As is well known, tones (musical or otherwise) cannot be completely represented by a single frequency. When the principal oboe offers an A, nominally at 440 Hz (or higher), the sound heard is a blend of that A with its natural overtones. The relative amplitudes ("loudnesses") of the various overtones are unique, enabling the oboe to sound like an oboe, rather than like a clarinet or a viola. Our ears identify that the A-440 (A4)[2] is, by far, the "loudest" of all of the component frequencies, and we have little difficulty making the approximation that we are hearing exactly one pitch. Although any naturally occurring sound is too complex to be adequately described with a single pitch assignment, most instruments project the fundamental pitch so predominantly that confusion is unlikely. (This article will continue to refer to musical tones by their fundamental pitches.)

The frequency of a pitch is usually measured in cycles (*i.e.* vibrational periods) per second, or Hertz (Hz). The overtones of any pitch are found by looking at all (integer) multiples of the fundamental frequency. The overtone series shown in Figure 1 is for a 100 Hz tone, which is close to G2, the open G-string on either a cello or a bass.[3] As with any overtone series, the seventh tone sounds significantly lower than the nearest chromatic pitch.

The overtone series also serves as a "library," so to speak, of consonant intervals. Pitches with a simple mathematical relationship to one another share overtones, a similarity that our ears identify as "consonance." A perfect fifth, for instance, is the interval between two pitches whose frequencies are in a 3:2 ratio; it is this sharing of overtones that allows string players to tune their

[2] The so-called scientific system of octave designation, with C4 equal to middle C, is used throughout.

[3] In an equal temperament system with A4 = 440 Hz, the frequency of this G is actually 98.0 Hz, but using 100 Hz will make the mathematical relationships more evident.

pairs of strings in perfect fifths with such ease. The traditional consonant intervals – perfect fourths and fifths, major and minor thirds and sixths – can each be represented by two notes of the same overtone series.

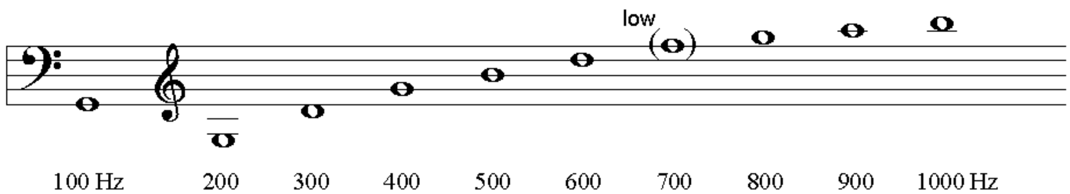

Figure 1. The overtone series for a 100 Hz tone (approximately G2)

The crucial piece of information to be understood here is this:

Fact #1.

Any interval can be represented by a frequency ratio; *each mathematical ratio is equivalent to exactly one musical interval.*

The simplest case is the octave. As can be seen above, the interval between a 200 Hz tone and a 100 Hz tone is an octave, and the two frequencies are in a 2:1 ratio.[4] But the interval between the 400 Hz tone (G3) and the 200 Hz tone (G4) is again an octave, with the same 2:1 relationship; similarly, the ratio between the 600 Hz tone (D4) and the 300 Hz tone (D5) is also an octave, still with the same 2:1 relationship. *Any* octave, under *any* circumstances, will correspond to a 2:1 frequency ratio. One can start with any arbitrary frequency and then double it, and the resulting pitch will be one octave above the original.

[4] This article will consistently interpret intervals as being descending, which results in "larger:smaller" ratios, but saying that an *ascending* octave is equivalent to a 1:2 ratio is also accurate.

The same principle is in effect for other intervals. A perfect fifth is a 3:2 ratio, as with the interval between the 300 Hz tone and the 200 Hz tone, but also as between the 900 Hz tone (A6)[5] and the 600 Hz tone (D6). A perfect fourth is found between the 400 Hz tone (G4) and the 300 Hz tone (D4), so a perfect fourth is a 4:3 ratio. By continuing in this fashion, we arrive at the frequency ratios that correspond to all consonant intervals, as shown in Table 2. Again, these frequency ratios correspond to the indicated intervals regardless of key or register. If ever any of the ratios are forgotten, writing out and examining any overtone series will be sufficient to bring the information back to mind.

interval	frequency ratio	earliest appearance in overtone series
Less than an octave		
perfect 5th	3:2	300 Hz (D4) and 200 Hz (G3)
perfect 4th	4:3	400 Hz (G4) and 300 Hz (D4)
major 3rd	5:4	500 Hz (B4) and 400 Hz (G4)
minor 3rd	6:5	600 Hz (D5) and 500 Hz (B4)
major 6th	5:3	500 Hz (B4) and 300 Hz (D4)
minor 6th	8:5	800 Hz (G5) and 500 Hz (B4)
More than an octave		
perfect 12th (octave + P5)	3:1	300 Hz (D4) and 100 Hz (G2)
perfect 11th (octave + P4)	8:3	800 Hz (G5) and 300 Hz (D4)
major 10th (octave + M3)	5:2	500 Hz (B4) and 200 Hz (G3)
two octaves	4:1	400 Hz (G4) and 100 Hz (G2)
two octaves + major 3rd	5:1	500 Hz (B4) and 100 Hz (G2)

Table 2. Frequency ratio equivalents for several consonant intervals.

It is worth reflecting on how powerful this realization is. In theory classes, most musicians are taught that a perfect fifth is the interval between the first and fifth degrees of a major or minor scale; perhaps later they come to think of a fifth as seven half-steps. Acoustically speaking,

[5] For A-440 tuning, this A5 is 880 Hz instead of 900 Hz, but recall that the 100 Hz “approximation” for G2 was too high, principally accounting for this discrepancy.

however, a perfect fifth can be *defined* as the interval between two pitches whose frequencies are in a 3:2 ratio. This association supercedes any consideration of register, tonality, or notation.[6]

3. String Length and Frequency

The frequency of a vibrating string depends on the string's density, tension, and length. Lower-pitched strings are denser than higher-pitched ones; the tension in the string is set by the player when he or she tunes the instrument. During the course of ordinary performance, the string's vibrating length is the only parameter that the performer modifies in order to influence the pitch.

When the string is played "open," its vibrating length is just the distance between the nut and the bridge, but when the performer uses a finger to stop the string, he or she shortens the string's vibrating length. The section of string between the nut and the finger ("behind" the finger) does not vibrate, as shown in Figure 2. In the diagram, the curved solid line represents the vibrating string, and the dashed line represents the string's position when not vibrating.

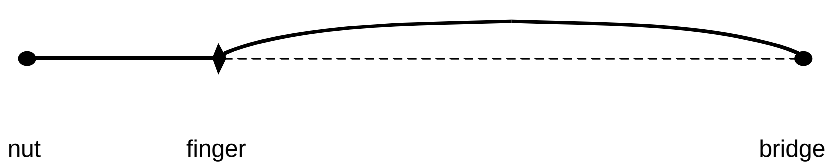

Figure 2. A vibrating string that is stopped by the performer's finger.

All musicians know this, but the following piece of information is less familiar:

Fact #2.

If the string is shortened to x/y *of its original length, its vibrating frequency becomes higher by a factor of* y/x.

[6] Tuning in equal temperament requires that intervals (other than the octave) be slightly different from their "pure" ratios. The perfect fifth, for instance, is a ratio of 2.9966:2, rather than 3:2. This nuance does not affect the discussion of harmonics, but a more thorough treatment of the issue is provided in the Appendix.

A few examples will help to clarify what may at first appear to be an “algebraic” formulation.[7]

- If the string is shortened to 2/3 of its original length, the new frequency is 3/2 as high as the original. If the open string had a vibrating frequency of 200 Hz, the shortened string will vibrate at 300 Hz (because 200 × 3/2 = 300).

- If the string is shortened to 3/5 of its original length, the new frequency is 5/3 as high as the original. If the open string had a vibrating frequency of 600 Hz, the shortened string will vibrate at 1000 Hz (because 600 × 5/3 = 1000).

- If the string is shortened to 1/4 of its original length, the new frequency is 4 times (i.e., 4/1) as high as the original. If the open string had a vibrating frequency of 150 Hz, the shortened string will vibrate at 600 Hz (because 150 × 4/1 = 600).

Note that, in all of these cases, we are interested in the length of string that is *still vibrating*, rather than that section of the string, behind the finger, that is motionless.

A useful concept: the fingerboard tape measure

Imagine a tape measure over the length of the fingerboard, scaled so that “0” is at the nut and “1” is at the bridge. At whatever point the finger stops the string, the remaining length of the string is what vibrates. Once we have this imaginary tape measure in mind, we can refer to positions on the fingerboard in solely mathematical terms, avoiding the restrictive association of finger position with pitch.

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[7] Those who are familiar with mathematical terminology will recognize this as simply, “frequency is inversely proportional to string length.”

As an example, a violinist playing "first position, third finger" is shortening the string so that it sounds a perfect fourth above the open string. What is the mathematical location of this finger position?

- A perfect fourth is equivalent to a ratio of 4:3 (see Figure 1 and Table 2 above).
- This means that the shortened string must vibrate at a frequency which is 4/3 as high as that of the open string.
- According to Fact #2, the shortened string will vibrate at this frequency if its length is 3/4 the length of the original.
- If 3/4 of the original length is vibrating, that means the remaining 1/4 is behind the finger.

We have now established that if the performer's finger stops the string at the 1/4 mark on our imaginary tape measure, the frequency of the vibrating string will be a perfect fourth higher than if the string were open (see Figure 3). This is the mathematical location of "first position, third finger." Again, we should reflect on this for a moment. Any length of string is subject to these considerations. If a finger or other device stops the string at the 1/4 point, so that 3/4 of the original length is vibrating, the resulting frequency will be 4/3 times the frequency of the open string, and the resulting pitch will be a perfect fourth above the pitch of the open string.

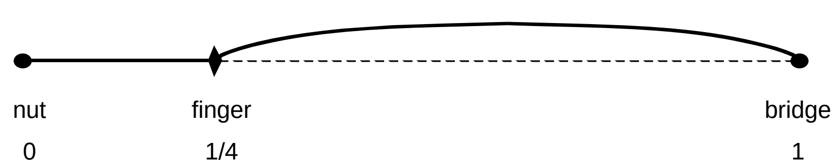

Figure 3. A stopped string sounding a perfect fourth above its open pitch.

A further example demonstrates how the same procedure works for a different interval. Imagine that the performer wishes to raise the pitch of the string by a major tenth.

- A major tenth is equivalent to a ratio of 5:2 (Figure 1 and Table 2).
- The shortened string must vibrate at a frequency 5/2 as high as the open string.
- The length of the shortened string must be 2/5 the length of the entire string (Fact #2).
- Since 2/5 of the string is vibrating, 3/5 of the string is behind the finger: the finger stops the string at the 3/5 point on the imaginary tape measure.

If this imaginary tape measure actually existed on string instruments, a less confusing notation for harmonics could be developed: the composer could simply indicate the mathematical point where the string is to be touched. Since no such tape measure exists, however, composers have no better option that the current system, identifying the point on the string by indicating what pitch *would* result if the string were to be stopped. When the composer needs a violinist to execute a harmonic at the 1/4 point on the A-string, the simplest resource is to refer to the point as "where the D is." While we can appreciate the reasons behind this system, we must also recognize that the pitch of the resulting harmonic depends on the 1/4 position, not on the note "D."

4. Natural harmonics

When the string is vibrating at a natural harmonic, its entire length is vibrating. The performer touches the string only enough to hold one point of it in place, but energy imparted by the bow propagates along the full length of the string, on both sides of the finger.

In the case of a stopped string, only the endpoints (at the bridge and the stopping finger) are fixed. When the string is being played at a natural harmonic, however, certain other points along the length of the string are also motionless. These points are called "nodes," and one of them is created by the touching finger. The possible vibrations of the string are limited by the following requirement:

Fact #3.

All nodes along the length of a vibrating string must be equally spaced between the endpoints.

This fact means that a vibration like the one represented in Figure 4a is not possible. Instead, if the performer's finger creates a node, the string will find an allowable vibration that accommodates the node, and other nodes will be created automatically. Since the nodes must be equally spaced, the vibration will effectively "split" the string into equal pieces, as shown in Figure 4b.[8] The string will vibrate at the lowest frequency that accommodates the node at the finger. If the performer's finger were at the 3/4 mark, instead of the 1/4 mark, the vibration – and thus the sounding harmonic – would be the same. However, if the performer's finger were at the 1/2 mark, the string would not vibrate as shown in Figure 4b, but would instead vibrate as shown in Figure 4c.

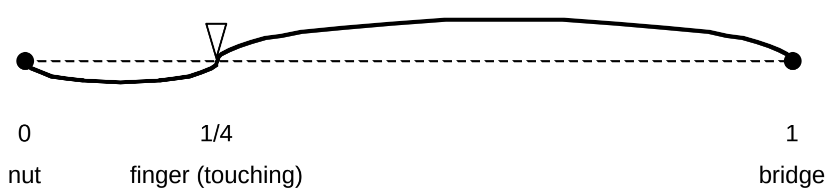

Figure 4a. A vibration that is not possible, because the nodes are not equally spaced.

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[8] One may wonder how the string "knows how" to jump up to a higher-frequency vibration. The answer is that, even when the string is open, its vibration is in fact very complex, simultaneously producing the fundamental and its overtones. When the performer's finger creates a node, all incompatible vibrations are "screened out," and only the harmonic – along with its own overtones – survives.

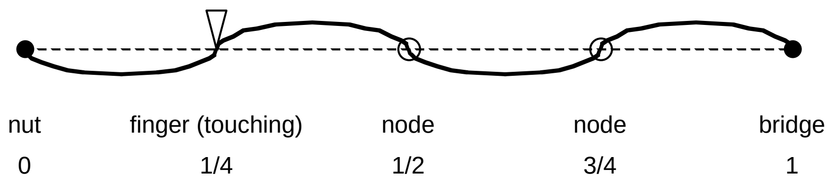

Figure 4b. The vibration created when the finger touches at the 1/4 point, with all nodes equally spaced.

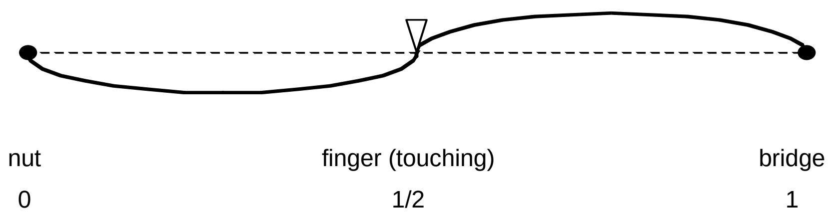

Figure 4c. The vibration created when the finger touches at the 1/2 point. Nodes at 1/4 and 3/4 are not created.

If required by the position of the finger, additional nodes will form on *both sides* of the finger at the same time. For instance, if the finger touches at the 2/5 point, the string will be split into five equal parts, and the additional nodes will be at 1/5, 3/5, and 4/5, as seen in Figure 5. The 1/5 point is behind the finger (between the finger and the nut), while the 3/5 and 4/5 points are between the finger and the bridge. Because no lower-frequency vibration is compatible with any of these nodes, the same harmonic will be generated by placing the finger at *any* of these four points.[9]

[9] The reader is encouraged to try this or to ask a string player for a demonstration. The four points in question are those that make the following intervals with the open string: major third, major sixth, major 10th, and major 17th (two octaves plus a major third).

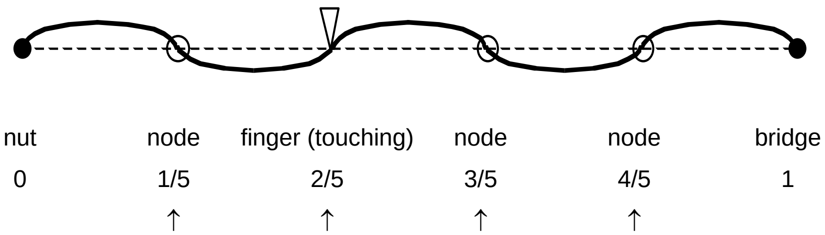

Figure 5. Touching the string at any of the points indicated by the arrows will yield the same harmonic.

To figure out what the lowest-frequency mode is for a particular finger placement, just look at the denominator of the finger's "position," remembering to put the fraction in lowest terms. For Figure 5, any position of the finger has the number "5" in the denominator, so the string must be split into five equal parts. But in Figures 4b and 4c, even though the middle point on the string could be considered as being at "2/4," the string will not split into four parts, because 2/4 can be reduced to 1/2. Some modes can be forced more easily than others can. Splitting the string into five parts can be accomplished with the finger at any of the possible positions indicated in Figure 5. To split the string into six parts, however, the finger must be placed at 1/6 or 5/6, because any other fraction can be reduced to lower terms: 2/6 = 1/3, 3/6 = 1/2, and so on.

If the string is split into *n* equal pieces, the frequency is the same as if only 1/*n* of the string were vibrating. All string players know that if the finger is at the midpoint of the string, the pitch is an octave higher than the open string; they also know that this pitch will result regardless of whether the string is stopped or played as a harmonic. If the string is stopped, only the half of its length ahead of the finger vibrates, whereas in the case of the harmonic, the entire string is vibrating with a node at the midpoint. Either way, however, the frequency corresponds to 1/2 of the string's length. The same principle, however, applies to higher harmonics. Returning to Figure 5, the frequency of the harmonic would be the same as if the string were *stopped* at the 4/5 point.

Fact #2 tells us that the vibrating frequency of 1/*n* of the string is *n*/1 times as high (in other words, *n* times as high) as the vibrating frequency of the full-length string. We therefore arrive at the following conclusion:

Fact #4

A natural harmonic that splits the string into n *equal parts results in a frequency that is* n *times as high as that of the open string.*

From this it follows that all natural harmonics must lie in the overtone series of the open string. The number *n* in Fact #4 can be thought of as a “frequency multiplier,” providing an easy way of referring to harmonics that also acknowledges the mathematical relationship. For instance, the harmonic at the octave has a frequency multiplier of 2, the harmonic at the perfect fifth (raising the pitch by an octave plus a perfect fifth) has a frequency multiplier of 3, and so on.[10]

We can now use our ability to think about points on the string “mathematically” as a tool to “convert” between stopped notes and harmonics. The stopped pitch and the harmonic pitch may or may not be closely related, but each is related to the position on the string as measured by the imaginary tape measure. Entirely unfamiliar harmonics can be figured out, step-by-step, based on these techniques.

As an example, imagine that a performer is asked to create a harmonic by touching the point a minor third above the open string. What pitch will result?

[10] In the traditional ordinal numbering, the “first harmonic” is at the 1/2 point, the “second harmonic” is at the 1/3 point, and so on; this counterintuitive system of identification can only guarantee confusion once the mechanism of harmonics is more fully understood. Brass players are familiar with “partials,” but many string players are not, which is why the term is avoided throughout this article.

• Step 1. *Represent the interval between the stopped note and the open string mathematically*. In the overtone series of Figure 1, a minor third appears between the 600 Hz and 500 Hz tones. A minor third is therefore equivalent to a 6:5 ratio.

• Step 2. *Identify the mathematical position of the finger*. Since the stopped string, when sounding a minor third above, has a frequency that is 6/5 as high as that of the open string, it must be true that 5/6 of the string is vibrating. This means that the remaining 1/6 of the string lies behind the finger, so the finger is at the 1/6 spot.

• Step 3. *Establish how the string will be split when the harmonic is played*. With the finger at the 1/6 position, the string will be split into 6 equal parts. The fraction 1/6 is already in lowest terms.

• Step 4. *Refer back to the overtone series to figure out the pitch of the harmonic*. With the string in 6 equal parts, the frequency of the harmonic is 6 times as high as that of the open string. A 6:1 frequency ratio is equivalent to two octaves and a perfect fifth, so the harmonic at the minor third creates a pitch that is two octaves plus a perfect fifth higher than the open string.

To figure out where the finger should be placed in order to achieve a particular harmonic, simply follow the above procedure backwards.

Successively higher harmonics can *theoretically* be found by touching at the points corresponding to 1/7, 1/8, 1/9, and so on. The problem is that these locations are so close together (especially on the violin and viola) that it is difficult for the finger to define the location of the node with the necessary degree of precision. The lower-frequency modes will always be more stable than the higher-frequency ones: for instance, getting the harmonic at the 1/6 point to speak is considerably more difficult than getting a good result from the harmonic at the 1/2 point. For these reasons, it is valuable to consider alternative locations for producing otherwise troublesome harmonics. For a harmonic with a "frequency multiplier" of *n*, a finger placement at any of the positions 1/*n*, 2/*n*, 3/*n*, *etc.* will produce the desired sound, as long as the fraction cannot be reduced to lower terms.

If a harmonic is required in the immediate context of other notes, the best fingering may be determined by considerations of hand position and so on. If the sole issue, however, is how best to get a harmonic to speak, then the most secure approach is to figure out which possible location is least likely to experience interference from other harmonics. This determination is made by establishing which mathematical position has the greatest distance from other harmonics – particularly from the lower-frequency harmonics, as they are more likely to cause interference. The "distances" are calculated by ordinary subtraction between the fractions, although the spacing is more immediately evident if the fractions are all converted to decimals. Table 3 gives the best positions for all harmonics up to frequency multiplier 10, which can be taken as a practical limit. Depending on the instrument, some of the harmonics below may be difficult to find; high-frequency harmonics in general require that the bow be very close to the bridge. Although this table could be memorized, such a chore is unnecessary. The computations required to reproduce this information are not complicated, and they rapidly become intuitive.

While it is true that some of the harmonics here are rarely, if ever, called for, the "standards" first listed in Table 1 are not sufficient for the standard repertoire. For example, Ravel, in his *L'enfant et les sortilèges* (Paris: Durand, 1925), asks for the harmonic at 3/8 (three octaves above the open string, notated at the minor sixth), in the third viola part starting one note before rehearsal number 49. The passage can easily lead to confusion if the players and conductor do not know what pitch is intended.

frequency multiplier	interval above open string (harmonic)	fractional position	decimal position	interval above open string (stopped)	comments
2	octave	1/2	0.500	octave	only option
3	octave + P5	1/3	0.333	P5	only option in lower half of string
4	2 octaves	1/4	0.250	P4	only option in lower half of string
5	2 octaves + M3	1/5	0.200	M3	generally fine, but occasional interference from 1/6
		2/5	0.400	M6	best
6	2 octaves + P5	1/6	0.167	m3	only option in lower half of string
7	2 octaves + m7 (low)	1/7	0.143	m3 (low)	too close to 1/6, not useable
		2/7	0.286	tritone (low)	interference from both 1/4 and 3/10, weak
		3/7	0.429	m7 (low)	interference from 4/9, but generally best
8	3 octaves	1/8	0.125	M2 (high)	too close to 1/7, not useable
		3/8	0.375	m6	best
9	3 octaves + M2	1/9	0.111	M2	too close to 1/8, not useable
		2/9	0.222	between M3 and P4	interference from both 1/4 and 1/5, weak
		4/9	0.444	m7 (high)	some interference from 3/7, but best
10	3 octaves + M3	1/10	0.100	–	not useable
		3/10	0.300	tritone (high)	interference from 2/7, but best

Table 3. Positions for natural harmonics up to frequency multiplier 10.

As with all musical elements, errors involving harmonics occasionally appear in scores and parts. Because of their quirky notation, harmonics can be written incorrectly in ways that do not immediately catch the eye. In the final movement (*Feria*) of Ravel's *Rhapsodie Espagnole*, the cellos are *divisi* two before rehearsal number 27 with the material shown in Figure 6a.[11] From the context of the passage (including woodwind doublings), it is obvious that Ravel intended the harmonics in the second part to double the first part at the octave. The harmonics indicated, however, generate the pitches shown on the right. The correct second cello part should be as given in Figure 6b.

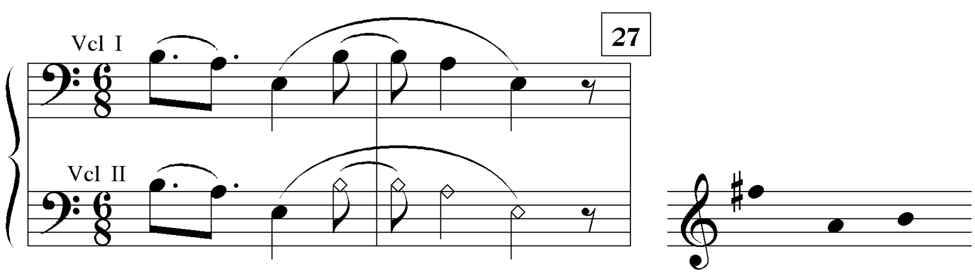

Figure 6a. The harmonics are intended to double the first part at the octave, but instead sound as the pitches on the right.

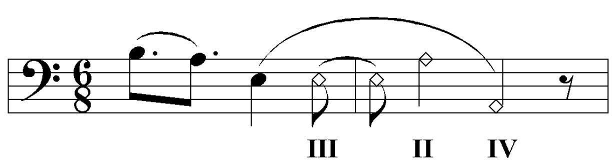

Figure 6b. The corrected second cello part for the passage in Figure 6a.

[11] Paris: Durand, 1908; reprint, with other works, New York: Dover, 1989. The error also appears in the orchestral parts available through Kalmus. The representation here is a simplification of what appears in the score, including the tacit omission of an additional error.

5. Artificial harmonics

When executing artificial harmonics, the performer uses two fingers (or a thumb and finger) for two different purposes. The finger closer to the nut closes ("stops") the string, and the finger closer to the bridge touches the string lightly, creating a node. The part of the string that is behind the stopping finger does not vibrate, but the remainder of the string does. Since natural harmonics are only available at pitches which lie in the overtone series of the open strings, artificial harmonics provide a way of achieving the distinctive color (or register) of harmonics with relative facility.

For figuring out artificial harmonics, it is important to keep in mind that the stopping finger (or thumb) has the effect of shortening the string's length, thereby creating a *new* overtone series. The touching finger, by creating a node, forces the now shorter string into a higher-frequency mode of vibration, where it will vibrate at a pitch that lies in the overtone series of the shortened string. For the most commonly used artificial harmonic, the interval between the stopping finger and the touching finger is a perfect fourth, resulting in a pitch two octaves above the stopped note. Figure 7 shows the case where the interval between the fingers is the usual perfect fourth and the *stopped* pitch is a perfect fifth above the open string.

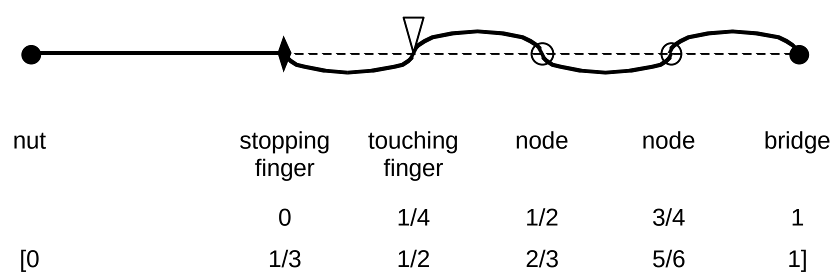

Figure 7. An artificial harmonic sounding two octaves and a fifth above the open string, which is two octaves above the shortened string created by the stopping finger.

While the two-octave harmonic is convenient, and most players are well versed in its execution, other artificial harmonics are possible, depending on the size of the performer's hand and the position on the string. As artificial harmonics do not speak as easily as natural harmonics, those requiring high-frequency modes of vibration may not be practical. (Indeed, with the flexibility provided by artificial harmonics, they should not be necessary.) Among the lower-frequency modes, however, it can be worthwhile to keep in mind the possible ways that a given pitch can be produced, since the context of the passage may make one fingering easier than others.

Example:

A cellist is asked to play G6 (fourth ledger line above the treble staff) as a harmonic. What are the options?

It is generally worthwhile to consider natural harmonics first. G6 lies in the natural overtone series of the open C- and G-strings, but these would require frequency multipliers of 24 and 16 respectively, neither of which is realistic. G6 lies only two octaves and a minor seventh above the open A-string, and is thus in the natural overtone series. This harmonic requires a frequency multiplier of 7, which is feasible, but it will be out of tune (see Table 3). G6 does not lie in the overtone series of the open D-string. Thus the prospects for a natural harmonic are exhausted, and an artificial harmonic is called for.

If the G6 is to be played as an artificial harmonic, we can identify candidate fundamental pitches according to the interval they make with G6. Thus:

- Two octaves below G6 is G4, so G4 is a candidate fundamental.
- One octave and a fifth below G6 (frequency multiplier 3) is C5.
- Two octaves and a major third below G6 (frequency multiplier 5) is E$^{\beta}$4.

For each possible fundamental pitch, the artificial harmonic is realized in the same way that a natural harmonic would be for the shortened string. Thus, referring back to Table 1 for convenience, the node-creating finger must touch the string at

- a perfect fourth above the G4, which is C5, or
- a perfect fifth above the C5, which is G5, or
- a major third above the E$^{\beta}$4, which is G4 (since the major sixth is too far to stretch)

Figure 8 shows the three possibilities as they would typically be notated.

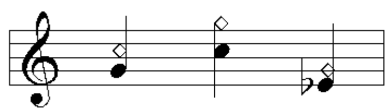

Figure 8. Three artificial harmonics that will yield the same pitch (G6).

Copland's *Concerto for Clarinet* (London: Boosey & Hawkes, 1952) demonstrates how a comprehensive knowledge of harmonics can enable the discovery of reasonable alternatives to awkward passages. The first violin part for measures 134 and 135 is shown in Figure 9a. The pitch of the harmonic is C$^{\beta}$7 (= B6). As written, the passage is clumsy. Substituting a natural harmonic on the E-string, as shown in Figure 9b, considerably simplifies the task.

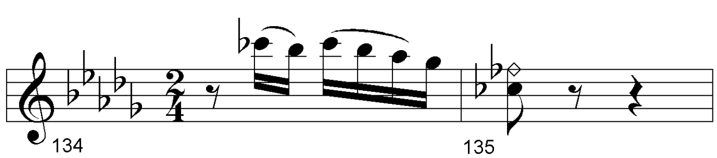

Figure 9a. Copland, *Concerto for Clarinet*, first violin part.

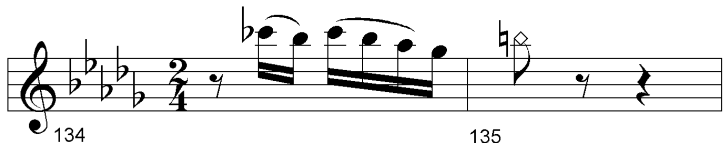

Figure 9b. The above passage simplified, using a natural harmonic on the E-string.

Toward the end of the piece, as shown in Figure 10a, the measures 498-500 present a more serious challenge for the first violins. As notated, the passage is impractical, especially in measure 500. Fortunately, a different fingering for the artificial harmonic (sounding $F^{\#}7$) is available, as shown in Figure 10b. Using this solution, the passage becomes manageable.

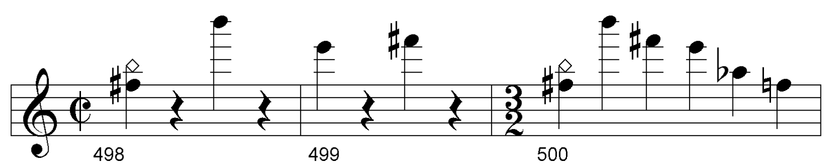

Figure 10a. Copland, *Concerto for Clarinet*, first violin part.

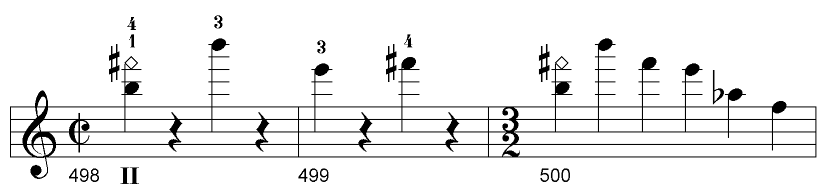

Figure 10b. A different artificial harmonic produces the same pitch.

Conclusion

An incomplete understanding of harmonics presents problems that can be exposed even by the standard repertoire, whereas a thorough knowledge can reveal printing errors or simplify tricky passages. Although the information presented here is not found in standard sources, it is essential for mastery of this commonly used resource. Mystery surrounding harmonics is unnecessary, as long as composers, conductors, and instrumentalists meet their responsibility to understand the physics of their production; less-experienced string players must know that they can rely on conductors to address such issues confidently and accurately. While the information to master may at first seem formidable, it all follows easily from a few elementary principles.

Appendix: A point of clarification with respect to equal temperament.

The “pure” intervals given in Table 2 are not compatible with equal temperament. In the modern system of equal temperament, with twelve exactly equal subdivisions of the octave, the perfect fifths are not actually 3:2 ratios, but are slightly smaller, at about 2.9966:2. Other intervals, besides the octave, are also slightly off from their pure ratios.

A clear example of the conflict between pure and tempered intervals is provided by a four-note augmented triad, like G – B – D$^{\#}$ – G. The necessity of an enharmonic spelling (the final G is more properly F$^{\aleph}$) is a clue that the simplest mathematical intervals will not suffice. Each interval of the triad is a major third, supposedly with a “pure” ratio of 5:4. Since each pitch in the triad has a frequency equal to 5/4 times the pitch below it, the “octave” spanned by the three major thirds can be represented by (5/4) × (5/4) × (5/4). This product is approximately 1.95, meaning that the frequency ratio between the upper and lower G’s is 1.95:1. As an octave, though, this figure must be precisely 2:1, not 1.95:1, which means that the system has broken down: the pure intervals are not adequate to handle even the modest chromatic harmony in this instance.

Theoretically, higher notes in the overtone series could be used to provide frequency ratios for less consonant intervals. However, as early as the 700 Hz note (F5), we find a pitch that sounds noticeably "out of tune" in an ordinary harmonic context. (The "pure" minor seventh, to the extent that it can even be called that, is lower than the equal-tempered minor seventh by more than 30 cents[12]; the difference is obvious during the solo bass harmonics at the beginning of Ravel's *L'enfant et les sortilèges*, in which the minor sevenths are "pure" in spite of the overall context of equal temperament.)

So, does a string player (or any other musician with control over his or her instrument's tuning) play equal-temperament intervals, or pure ones? The answer varies, of course, but the issue is one of the reasons that string quartets (for instance) must spend so much time on intonation – the "best" pitch for any given note depends subtly on the circumstances, and the decision can legitimately be subjective. The well-known rule of thumb to play the third of a major triad slightly lower than usual stems provides a simple example: the major third in equal temperament is wider than a pure major third, and the difference becomes readily apparent when trying to tune a major triad.

The miracle of an equally tempered scale in only 12 subdivisions of the octave is that it comes very close to accommodating all of the consonant intervals listed in Table 2. The worst approximation is the minor third (or major sixth), where the discrepancy between the pure and equal-tempered representations of the interval is about 16 cents, or very roughly one-sixth of a semitone. While this difference is easily detected by a trained ear, the music-listening population has, by and large, been able to adjust to it.

As important as these nuances are to the general issue of tuning, they do not affect the thrust of this article, but are instead included for the sake of completeness.

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[12] A "cent" is one one-hundredth of a semitone, *i.e.* an interval corresponding to the ratio $2^{(1/1200)}$:1.